fields, and how the bound-charge current affects the "curl **B**" Maxwell equation. Finally, we consider an electromagnetic wave in a dielectric. We find that only a slight modification to the vacuum case is needed.

10.1 Dielectrics

The capacitor we studied in Chapter 3 consisted of two conductors, insulated from one another, with nothing in between. The system of two conductors was characterized by a certain capacitance C, a constant relating the magnitude of the charge Q on the capacitor (positive charge Q on one plate, equal negative charge on the other) to the difference in electric potential between the two conductors, $\phi_1 - \phi_2$. Let's denote the potential difference by ϕ_{12}:

$$C = \frac{Q}{\phi_{12}}. \tag{10.1}$$

For the parallel-plate capacitor, two flat plates each of area A and separated by a distance s, we found that the capacitance is given by

$$C = \frac{\epsilon_0 A}{s}. \tag{10.2}$$

Capacitors like this can be found in some electrical apparatus. They are called *vacuum capacitors* and consist of plates enclosed in a highly evacuated bottle. They are used chiefly where extremely high and rapidly varying potentials are involved. Far more common, however, are capacitors in which the space between the plates is filled with some nonconducting solid or liquid substance. Most of the capacitors you have worked with in the laboratory are of that sort; there are dozens of them in any television screen. For conductors embedded in a material medium, Eq. (10.2) does not agree with experiment. Suppose we fill the space between the two plates shown in Fig. 10.1(a) with a slab of plastic, as in Fig. 10.1(b). Experimenting with this new capacitor, we still find a simple proportionality between charge and potential difference, so that we can still *define* a capacitance by Eq. (10.1). But we find C to be substantially *larger* than Eq. (10.2) would have predicted. That is, we find more charge on each of the plates, for the same potential difference, plate area, and distance of separation. The plastic slab must be the cause of this.

It is not hard to understand in a general way how this comes about. The plastic slab consists of molecules, the molecules are composed of atoms, which in turn are made of electrically charged particles – electrons and atomic nuclei. The electric field between the capacitor plates acts on those charges, pulling the negative charges up, if the upper plate is positive as in Fig. 10.2, and pushing the positive charges down. Nothing moves very far. (There are no free electrons around, already detached

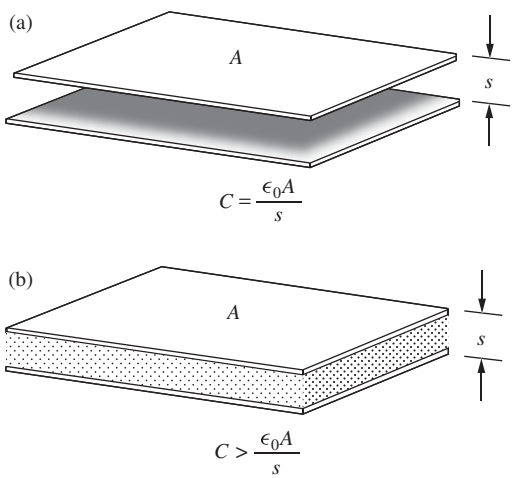

Figure 10.1.
(a) A capacitor formed by parallel conducting plates. (b) The same plates with a slab of insulator in between.

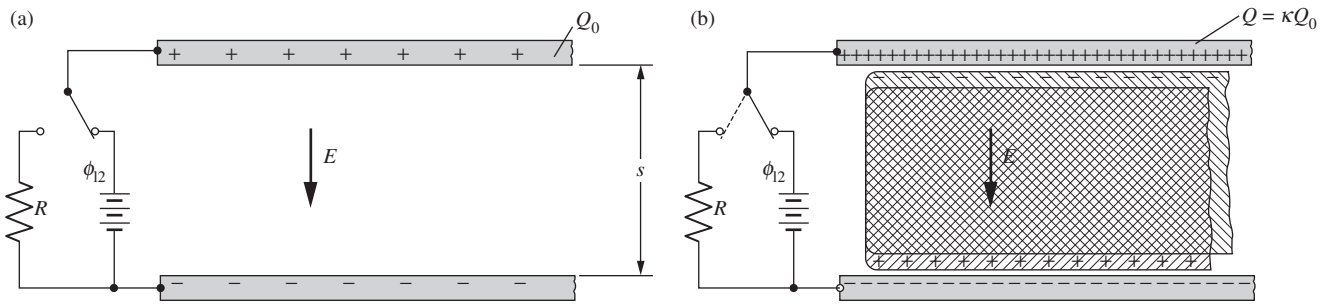

Figure 10.2.
How a dielectric increases the charge on the plates of a capacitor. (a) Space between the plates empty; $Q_0 = C_0\phi_{12}$. (b) Space between the plates filled with a nonconducting material, that is, a dielectric. Electric field pulls negative charges up and pushes positive charges down, exposing a layer of uncompensated negative charge on the upper surface of the dielectric and a layer of uncompensated positive charge on the lower surface. The total charge at the top, *including* charge Q on the upper plate, is the same as in (a). Q itself is now greater than Q_0; $Q = \kappa Q_0$. This Q is the amount of charge that will flow through the resistor R if the capacitor is discharged by throwing the switch.

from atoms and ready to travel, as there would be in a metallic conductor.) There will be some slight displacement of the charges nevertheless, for an atom is not an infinitely rigid structure. The effect of this within the plastic slab is that the negative charge distribution, viewed as a whole, and the total positive charge distribution (the atomic nuclei) are very slightly displaced relative to one another, as indicated in Fig. 10.2(b). The interior of the block remains electrically neutral, but a thin layer of uncompensated negative charge has emerged at the top, with a corresponding layer of uncompensated positive charge at the bottom.

In the presence of the induced layer of negative charge below the upper plate, the charge Q on the plate itself will increase. In fact, Q must increase until the total charge at the top, the algebraic sum of Q and the induced charge layer, equals Q_0 (the charge on the upper plate before the plastic was inserted). We shall be able to prove this when we return to this problem in Section 10.8 after settling some questions about the electric field inside matter. The important point now is that the charge Q in Fig. 10.2(b) is *larger* than Q_0 and that this Q *is* the charge of the capacitor in the relation $Q = C\phi_{12}$. It is the charge that came out of the battery, and it is the amount of charge that would flow through the resistor R were we to discharge the capacitor by throwing the switch in the diagram. If we did that, the induced charge layer, which is not part of Q, would simply disappear into the slab.

According to this explanation, the ability of a particular material to increase the capacitance ought to depend on the amount of electric charge in its structure and the ease with which the electrons can be displaced with respect to the atomic nuclei. The factor by which the capacitance is increased when an empty capacitor is filled with a particular material, Q/Q_0 in our example, is called the *dielectric constant* of that material. The symbol κ is usually used for it:

$$\boxed{Q = \kappa Q_0} \quad \Longleftrightarrow \quad \boxed{C = \kappa C_0} \tag{10.3}$$

The material itself is often called a *dielectric* when we are talking about its behavior in an electric field. But any homogeneous nonconducting

Table 10.1.
Dielectric constants of various substances

Substance	Conditions	Dielectric constant (κ)
Air	gas, 0 °C, 1 atm	1.00059
Methane, CH_4	gas, 0 °C, 1 atm	1.00088
Hydrogen chloride, HCl	gas, 0 °C, 1 atm	1.0046
Water, H_2O	gas, 110 °C, 1 atm	1.0126
	liquid, 20 °C	80.4
Benzene, C_6H_6	liquid, 20 °C	2.28
Methanol, CH_3OH	liquid, 20 °C	33.6
Ammonia, NH_3	liquid, −34 °C	22.6
Mineral oil	liquid, 20 °C	2.24
Sodium chloride, NaCl	solid, 20 °C	6.12
Sulfur, S	solid, 20 °C	4.0
Silicon, Si	solid, 20 °C	11.7
Polyethylene	solid, 20 °C	2.25–2.3
Porcelain	solid, 20 °C	6.0-8.0
Paraffin wax	solid, 20 °C	2.1–2.5
Pyrex glass 7070	solid, 20 °C	4.00

substance can be so characterized. Table 10.1 lists the measured values of the dielectric constants for a miscellaneous assortment of substances.

Every dielectric constant in the table is larger than 1. We should expect that if our explanation is correct. The presence of a dielectric could *reduce* the capacitance below that of the empty capacitor only if its electrons moved, when the electric field was applied, in a direction opposite to the resulting force. For oscillating electric fields, by the way, some such behavior would not be absurd. But for the steady fields we are considering here it can't work that way.

The dielectric constant of a perfect vacuum is, of course, exactly 1.0 by our definition. For gases under ordinary conditions, κ is only a little larger than 1.0, simply because a gas is mostly empty space. Ordinary solids and liquids usually have dielectric constants ranging from 2 to 6 or so. Note, however, that liquid ammonia is an exception to this rule, and water is a spectacular exception. Actually liquid water is slightly conductive, but that, as we shall have to explain later, does not prevent our defining and measuring its dielectric constant. The ionic conductivity of the liquid is not the reason for the gigantic dielectric constant of water. You can discern this extraordinary property of water in the dielectric constant of the vapor if you remember that it is really the *difference* between κ and 1 that reveals the electrical influence of the material. Compare the values of κ given in the table for water vapor and for air.

Once the dielectric constant of a particular material has been determined, perhaps by measuring the capacitance of one capacitor filled with it, we are able to predict the behavior, not merely of two-plate capacitors, but of *any* electrostatic system made up of conductors and pieces of that

dielectric of any shape. That is, we can predict all electric fields that will exist in the vacuum outside the dielectrics for given charges or potentials on the conductors in the system.

The theory that enables us to do this was fully worked out by the physicists of the nineteenth century. Lacking a complete picture of the atomic structure of matter, they were more or less obliged to adopt a macroscopic description. From that point of view, the interior of a dielectric is a featureless expanse of perfectly smooth "mathematical jelly" whose single electrical property distinguishing it from a vacuum is a dielectric constant different from unity.

If we develop only a macroscopic description of matter in an electric field, we shall find it hard to answer some rather obvious-sounding questions – or, rather, hard to ask these questions in such a way that they can be meaningfully answered. For instance, what is the strength of the electric field *inside* the plastic slab of Fig. 10.1(b) when there are certain charges on the plates? Electric field strength is defined by the force on a test charge. How can we put a test charge inside a perfectly dense solid, without disturbing anything, and measure the force on it? What would that force mean if we did measure it? You might think of boring a hole and putting the test charge in the hole with some room to move around, so that you can measure the force on it as on a free particle. But then you will be measuring not the electric field in the dielectric, but the electric field in a cavity in the dielectric, which is quite a different thing.

Fortunately another line of attack is available to us, one that leads up from the microscopic or *atomic* level. We know that matter is made of atoms and molecules; these in turn are composed of elementary charged particles. We know something about the size and structure of these atoms, and we know something about their arrangement in crystals and fluids and gases. Instead of describing our dielectric slab as a volume of structureless but nonvacuous jelly, we shall describe it as a collection of molecules inhabiting a vacuum. If we can find out what the electric charges in *one* molecule do when that molecule is all by itself in an electric field, we should be able to understand the behavior of two such molecules a certain distance apart in a vacuum. It will only be necessary to include the influence, on each molecule, of any electric field arising from the other. This is a vacuum problem. Now all we have to do is extend this to a population of, say, 10^{20} molecules occupying a cubic centimeter or so of vacuum, and we have our real dielectric. We hope to do this without generating 10^{20} separate problems.

This program, if carried through, will reward us in two ways. We shall be able at last to say something meaningful about the electric and magnetic fields inside matter, answering questions such as the one raised above. What is more valuable, we shall understand how the macroscopic electric and magnetic phenomena in matter arise from, and therefore reveal, the nature of the underlying atomic structure. We are going to study electric and magnetic effects separately. We begin with dielectrics.

Since our first goal is to describe the electric field produced by an atom or molecule, it will help to make some general observations about the electrostatic field external to any small system of charges.

10.2 The moments of a charge distribution

An atom or molecule consists of some electric charges occupying a small volume, perhaps a few cubic angstroms ($10^{-30}\,\mathrm{m}^3$) of space. We are interested in the electric field outside that volume, which arises from this rather complicated charge distribution. We shall be particularly concerned with the field far away from the source, by which we mean far away compared with the size of the source itself. What features of the charge structure mainly determine the field at remote points? To answer this, let's look at some arbitrary distribution of charges and see how we might go about computing the field at a point outside it. The discussion in this and the following section generalizes our earlier discussion of dipoles in Section 2.7.

Figure 10.3 shows a charge distribution of some sort located in the neighborhood of the origin of coordinates. It might be a molecule consisting of several positive nuclei and quite a large number of electrons. In any case we shall suppose it is described by a given charge density function $\rho(x, y, z)$; ρ is negative where the electrons are and positive where the nuclei are. To find the electric field at distant points we can begin by computing the potential of the charge distribution. To illustrate, let's take some point A out on the z axis. (Since we are not assuming any special symmetry in the charge distribution, there is nothing special about the z axis.) Let r be the distance of A from the origin. The electric potential at A, denoted by ϕ_A, is obtained as usual by adding the contributions from all elements of the charge distribution:

$$\phi_A = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x', y', z')\,dv'}{R}. \tag{10.4}$$

In the integrand, dv' is an element of volume within the charge distribution, $\rho(x', y', z')$ is the charge density there, and R in the denominator is the distance from A to this particular charge element. The integration is carried out in the coordinates x', y', z', of course, and is extended over all the region containing charge. We can express R in terms of r and the distance r' from the origin to the charge element. Using the law of cosines with θ the angle between $\mathbf{r}'$ and the axis on which A lies, we have

$$R = (r^2 + r'^2 - 2rr'\cos\theta)^{1/2}. \tag{10.5}$$

With this substitution for R, the integral becomes

$$\phi_A = \frac{1}{4\pi\epsilon_0}\int \rho\,dv'(r^2 + r'^2 - 2rr'\cos\theta)^{-1/2}. \tag{10.6}$$

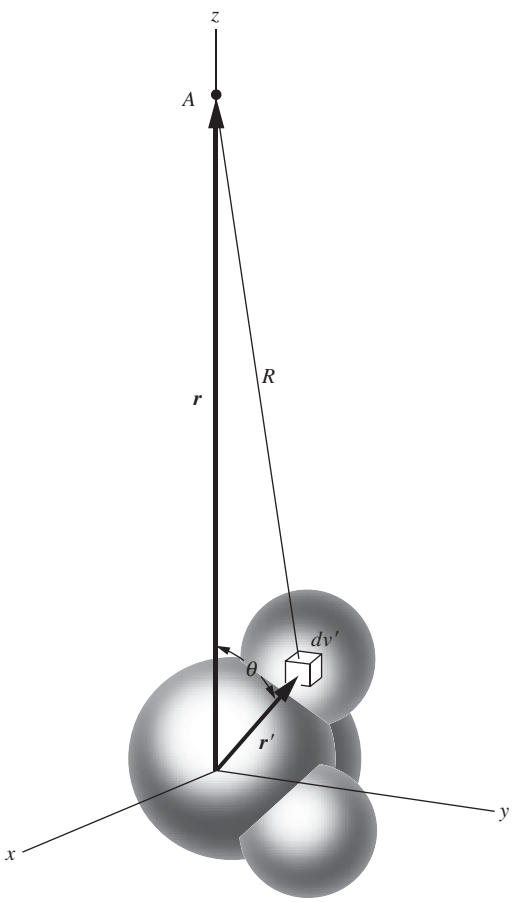

Figure 10.3.
Calculation of the potential, at a point A, of a molecular charge distribution.

10.5 Atomic and molecular dipoles; induced dipole moments

Consider the simplest atom, the hydrogen atom, which consists of a nucleus and one electron. If you imagine the negatively charged electron revolving around the positive nucleus like a planet around the sun – as in the original atomic model of Niels Bohr – you will conclude that the atom has, at any one instant of time, an electric dipole moment. The dipole moment vector **p** points parallel to the electron–proton radius vector, and its magnitude is e times the electron–proton distance. The direction of this vector will be continually changing as the electron, in this picture of the atom, circles around its orbit. To be sure, the *time average* of **p** will be zero for a circular orbit, but we should expect the periodically changing dipole moment components to generate rapidly oscillating electric fields and electromagnetic radiation.

The absence of such radiation in the normal hydrogen atom was one of the baffling paradoxes of early quantum physics. Modern quantum mechanics tells us that it is better to think of the hydrogen atom in its lowest energy state (the usual condition of most of the hydrogen atoms in the universe) as a spherically symmetrical structure with the electronic charge distributed, in the time average, over a cloud surrounding the nucleus. Nothing is revolving in a circle or oscillating. If we could take a snapshot with an exposure time shorter than 10^{-16} s, we might discern an electron localized some distance away from the nucleus. But for processes involving times much longer than that, we have, in effect, a smooth distribution of negative charge surrounding the nucleus and extending out in all directions with steadily decreasing density. The total charge in this distribution is just $-e$, the charge of one electron. Roughly half of it lies within a sphere of radius 0.5 angstrom ($0.5 \cdot 10^{-10}$ m). The density decreases exponentially outward; a sphere only 2.2 angstroms in radius contains 99 percent of the charge. The electric field in the atom is just what a stationary charge distribution of this form, together with the positive nucleus, would produce.

A similar picture is the best one to adopt for other atoms and molecules. We can treat the nuclei in molecules as point charges; for our present purposes their size is too small to matter. The entire electronic structure of the molecule is to be pictured as a single cloud of negative charge of smoothly varying density. The shape of this cloud, and the variation of charge density within it, will of course be different for different molecules. But at the fringes of the cloud the density will always fall off exponentially, so that it makes some sense to talk of the size and shape of the molecular charge distribution.

Quantum mechanics makes a crucial distinction between stationary states and time-dependent states of an atom. The state of lowest energy is a time-independent structure, a stationary state. It has to be, according to the laws of quantum mechanics. It is that state of the atom or

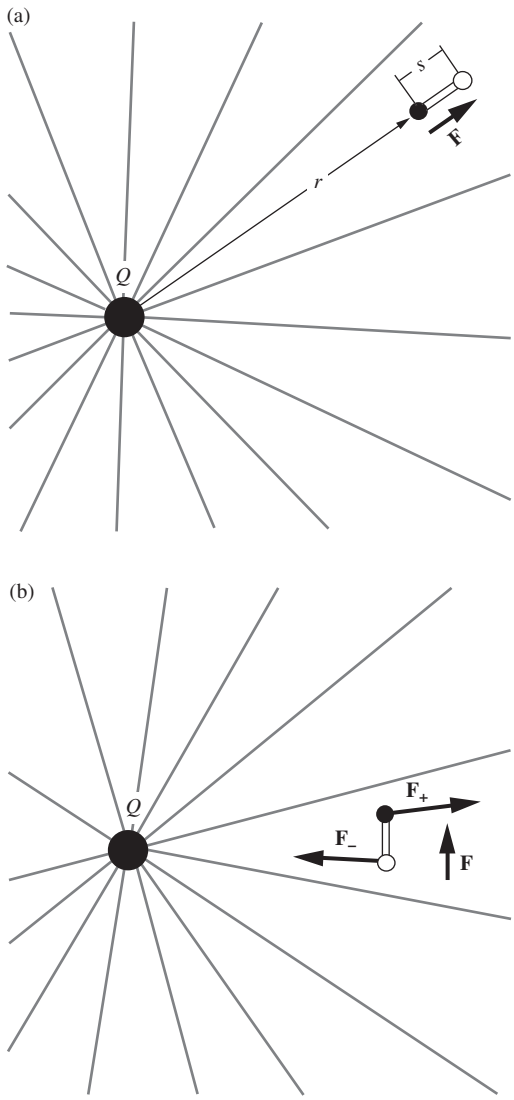

Figure 10.9.
The force on a dipole in a nonuniform field. (a) The net force on the dipole in this position is radially outward. (b) The net force on the dipole in this position is upward.

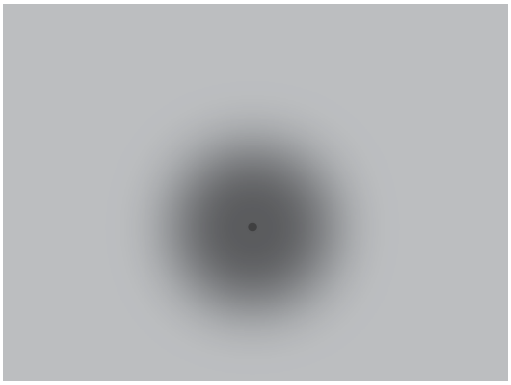

Figure 10.10.
The time-average distribution in the normal hydrogen atom. Shading represents density of electronic (negative) charge.

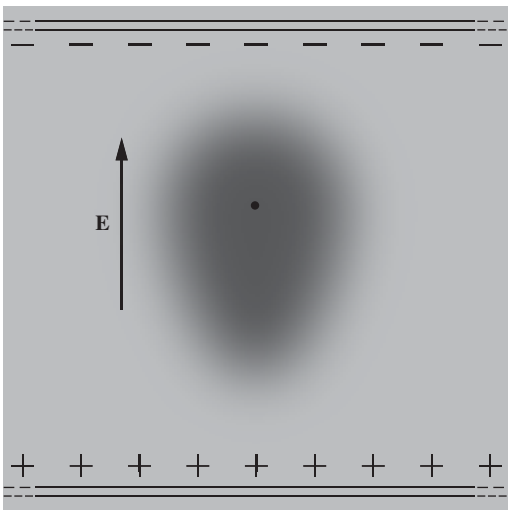

Figure 10.11.
In an electric field, the negative charge is pulled one way and the positive nucleus is pulled the other way. The distortion is grossly exaggerated in this picture. To distort the atom that much would require a field of 10^{10} volts/m.

molecule that concerns us here. Of course, atoms *can* radiate electromagnetic energy. That happens with the atom in a nonstationary state in which there is an oscillating electric charge.

Figure 10.10 represents the charge distribution in the normal hydrogen atom. It is a cross section through the spherically symmetrical cloud, with the density suggested by shading. Obviously the dipole moment of such a distribution is zero. The same is true of any atom in its state of lowest energy, no matter how many electrons it contains, for in all such states the electron distribution has spherical symmetry. It is also true of any ionized atom, though an ion of course has a monopole moment, that is, a net charge.

So far we have found nothing very interesting. But now let us put the hydrogen atom in an electric field supplied by some external source, as in Fig. 10.11. The electric field distorts the atom, pulling the negative charge down and pushing the positive nucleus up. The distorted atom will have an electric dipole moment because the "center of gravity" of the negative charge will no longer coincide with the positive nucleus, but will be displaced from the nucleus by some small distance Δz. The electric dipole moment of the atom is now $e\,\Delta z$.

How much distortion will be caused by a field of given strength E? Remember that electric fields already exist in the unperturbed atom, of strength $e/4\pi\epsilon_0 a^2$ in order of magnitude, where a is a typical atomic dimension. We should expect the relative distortion of the atom's structure, measured by the ratio $\Delta z/a$, to have the same order of magnitude as the ratio of the perturbing field E to the internal fields that hold the atom together. We predict, in other words, that

$$\frac{\Delta z}{a} \approx \frac{E}{e/4\pi\epsilon_0 a^2}. \tag{10.27}$$

If you don't trust this reasoning, Exercise 10.30 gives an alternative method for finding the relation between Δz and E.

Now a is a length of order 10^{-10} m, and $e/4\pi\epsilon_0 a^2$ is approximately 10^{11} volts/m, a field thousands of times more intense than any large-scale steady field we could make in the laboratory. Evidently the distortion of the atom is going to be very slight indeed, in any practical case. If Eq. (10.27) is correct, it follows that the dipole moment p of the distorted atom, which is just $e\,\Delta z$, will be

$$p = e\,\Delta z \approx 4\pi\epsilon_0 a^3 E. \tag{10.28}$$

Since the atom was spherically symmetrical before the field $\mathbf{E}$ was applied, the dipole moment vector $\mathbf{p}$ will be in the direction of $\mathbf{E}$. The factor that relates $\mathbf{p}$ to $\mathbf{E}$ is called the *atomic polarizability,* and is usually denoted by α:

$$\boxed{\mathbf{p} = \alpha\mathbf{E}} \tag{10.29}$$

Table 10.2.
Atomic polarizabilities ($\alpha/4\pi\epsilon_0$), in units of $10^{-30}\,\mathrm{m}^3$

Element	H	He	Li	Be	C	Ne	Na	Ar	K
$\alpha/4\pi\epsilon_0$	0.66	0.21	12	9.3	1.5	0.4	27	1.6	34

It is common to work instead with the quantity $\alpha/4\pi\epsilon_0$, which has the dimensions of volume. The reason for this is that a direct comparison between $\mathbf{p}$ and $\mathbf{E}$ isn't quite a fair one, because electric fields contain a somewhat arbitrary factor of $1/4\pi\epsilon_0$ multiplying the factors of charge and distance in Coulomb's law. A more reasonable comparison would therefore involve $\mathbf{p}$ and $4\pi\epsilon_0\mathbf{E}$. These quantities have dimensions of (charge) $\times$ (distance) and (charge)/(distance)^2, respectively. Equation (10.29) then yields $\mathbf{p}/(4\pi\epsilon_0\mathbf{E}) = \alpha/4\pi\epsilon_0$. This quantity is often also called the atomic polarizability, so the term is a little ambiguous. It is best to say explicitly whether you are working with α or $\alpha/4\pi\epsilon_0$.

According to our estimate in Eq. (10.28), we have $\alpha \approx 4\pi\epsilon_0 a^3$, so $\alpha/4\pi\epsilon_0$ is in order of magnitude an atomic volume, something like $a^3 \approx 10^{-30}\,\mathrm{m}^3$. Its value for a particular atom will depend on the details of the atom's electronic structure. An exact quantum-mechanical calculation of the polarizability of the hydrogen atom predicts $\alpha/4\pi\epsilon_0 = (9/2)a_0^3$, where a_0 is the *Bohr radius,* $0.52 \cdot 10^{-10}$ m, the characteristic distance in the H-atom structure in its normal state. The values of $\alpha/4\pi\epsilon_0$ for several species of atoms, experimentally determined, are given in Table 10.2. The examples given are arranged in order of increasing number of electrons. Note the wide variations in $\alpha/4\pi\epsilon_0$. If you are acquainted with the periodic table of the elements, you may discern something systematic here. Hydrogen and the alkali metals lithium, sodium, and potassium, which occupy the first column of the periodic table, have large values of $\alpha/4\pi\epsilon_0$, and these increase steadily with increasing atomic number, from hydrogen to potassium. The noble gases have much smaller atomic polarizabilities, but these also increase as we proceed, within the family, from helium to neon to krypton. Apparently the alkali atoms, as a class, are easily deformed by an electric field, whereas the electronic structure of a noble gas atom is much stiffer. It is the loosely bound outer, or "valence," electron in the alkali atom structure that is responsible for the easy polarizability.

A molecule, too, develops an induced dipole moment when an electric field is applied to it. The methane molecule depicted in Fig. 10.12 is made from four hydrogen atoms arranged at the corners of a tetrahedron around the central carbon atom. This object has an electrical polarizability, determined experimentally, of

$$\frac{\alpha}{4\pi\epsilon_0} = 2.6 \cdot 10^{-30}\,\mathrm{m}^3. \tag{10.30}$$

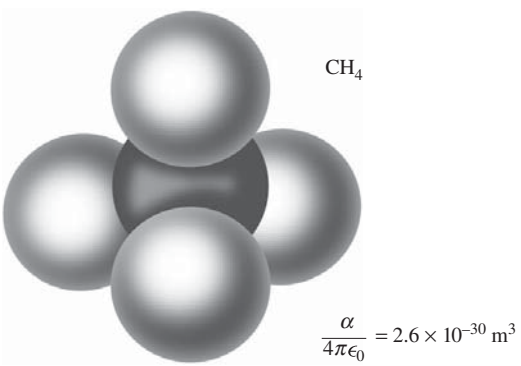

Figure 10.12.
The methane molecule, made of four hydrogen atoms and a carbon atom.

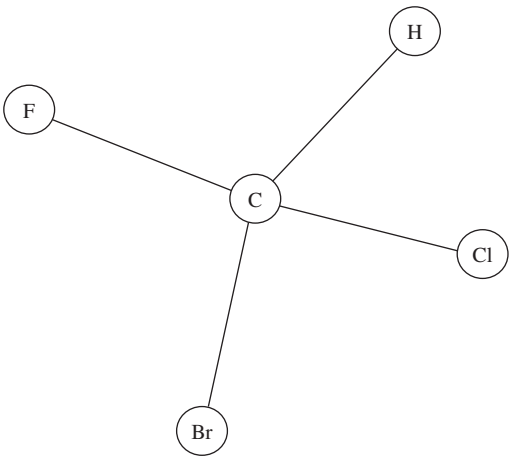

Figure 10.13.
A molecule with no symmetry whatsoever, bromochloroflouromethane. This is methane with three different halogens substituted for three of the hydrogens. The bond lengths and the tetrahedron edges are all a bit different.

It is interesting to compare this with the sum of the polarizabilities of a carbon atom and four isolated hydrogen atoms. Taking the data from Table 10.2, we find $\alpha_C/4\pi\epsilon_0 + 4\alpha_H/4\pi\epsilon_0 = 4.1 \cdot 10^{-30}\,\mathrm{m}^3$. Evidently the binding of the atoms into a molecule has somewhat altered the electronic structure. Measurements of atomic and molecular polarizabilities have long been used by chemists as clues to molecular structure.

10.6 Permanent dipole moments

Some molecules are so constructed that they have electric dipole moments even in the absence of an electric field. They are unsymmetrical in their normal state. The molecule shown in Fig. 10.13 is an example. A simpler example is provided by any diatomic molecule made out of dissimilar atoms, such as hydrogen chloride, HCl. There is no point on the axis of this molecule about which the molecule is symmetrical fore and aft; the two ends of the molecule are physically different. It would be a pure accident if the center of gravity of the positive charge and that of the negative charge happened to fall at the same point along the axis. When the HCl molecule is formed from the originally spherical H and Cl atoms, the electron of the H atom shifts partially over to the Cl structure, leaving the hydrogen nucleus partially denuded. So there is some excess of positive charge at the hydrogen end of the molecule and a corresponding excess of negative charge at the chlorine end. The magnitude of the resulting electric dipole moment, $p = 3.4 \cdot 10^{-30}$ coulomb-meter, is equivalent to shifting one electron about 0.2 angstrom (using $s = p/e$).

By contrast, the hydrogen atom in a field of 1 megavolt per meter, with the polarizability listed in Table 10.2, acquires an induced moment less than 10^{-34} coulomb-meter. Permanent dipole moments, when they exist, are as a rule enormously larger than any moment that can be induced by ordinary laboratory electric fields.[2] Because of this, the distinction between *polar* molecules, as molecules with "built-in" dipole moments are called, and *nonpolar* molecules is very sharp.

We said at the beginning of Section 10.5 that the hydrogen atom had, at any instant of time, a dipole moment. But then we dismissed it as being zero in the time average, on account of the rapid motion of the electron. Now we seem to be talking about molecular dipole moments as if a molecule were an ordinary stationary object like a baseball bat whose ends could be examined at leisure to see which was larger! Molecules move more slowly than electrons, but their motion is rapid by ordinary standards. Why can we credit them with "permanent" electric dipole moments? If this inconsistency was bothering you, you are to be commended. The full answer can't be given without some quantum

[2] There is a good reason for this. The internal electric fields in atoms and molecules, as we remarked in Section 10.5, are naturally on the order of $e/4\pi\epsilon_0(10^{-10}\,\mathrm{m})^2$, which is roughly 10^{11} volts/m! We cannot apply such a field to matter in the laboratory for the closely related reason that it would tear the matter to bits.

mechanics, but the difference essentially involves the time scale of the motion. The time it takes a molecule to interact with its surroundings is generally *shorter* than the time it takes the intrinsic motion of the molecule to average out the dipole moment smoothly. Hence the molecule *really acts* as if it had the moment we have been talking about. A very short time qualifies as permanent in the world of one molecule and its neighbors.

Some common polar molecules are shown in Fig. 10.14, with the direction and magnitude of the permanent dipole moment indicated for each. The water molecule has an electric dipole moment because it is bent in the middle, the O–H axes making an angle of about 105° with one another. This is a structural oddity with the most far-reaching consequences. The dipole moment of the molecule is largely responsible for the properties of water as a solvent, and it plays a decisive role in chemistry that goes on in an aqueous environment. It is hard to imagine what the world would be like if the H_2O molecule, like the CO_2 molecule, had its parts arranged in a straight line; probably we wouldn't be here to observe it. We hasten to add that the shape of the H_2O molecule is not a capricious whim of Nature. Quantum mechanics has revealed clearly why a molecule made of an eight-electron atom joined to two one-electron atoms must prefer to be bent.

The behavior of a polar substance as a dielectric is strikingly different from that of material composed of nonpolar molecules. The dielectric constant of water is about 80, that of methyl alcohol 33, while a typical nonpolar liquid might have a dielectric constant around 2. In a nonpolar substance the application of an electric field induces a slight dipole moment in each molecule. In the polar substance dipoles are already present in great strength but, in the absence of a field, are pointing in random directions so that they have no large-scale effect. An applied electric field merely *aligns* them to a certain degree. In either process, however, the macroscopic effects will be determined by the net amount of polarization per unit volume.

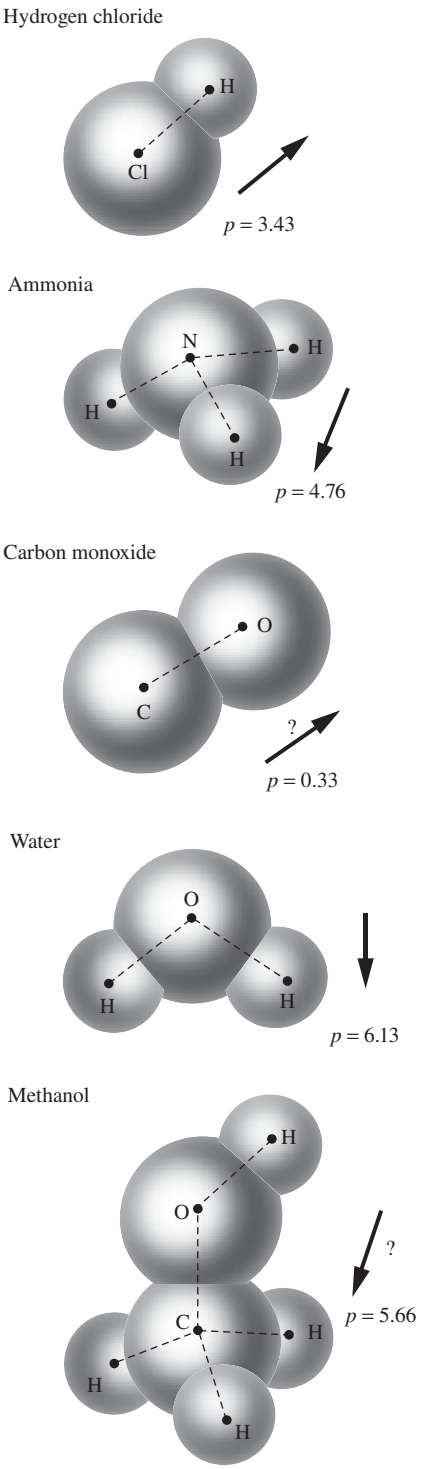

Figure 10.14.
Some well-known polar molecules. The observed value of the permanent dipole moment p is given in units of 10^{-30} coulomb-meters.

10.7 The electric field caused by polarized matter

10.7.1 The field outside matter

Suppose we build up a block of matter by assembling a very large number of molecules in a previously empty region of space. Suppose too that each of these molecules is polarized in the same direction. For the present we need not concern ourselves with the nature of the molecules or with the means by which their polarization is maintained. We are

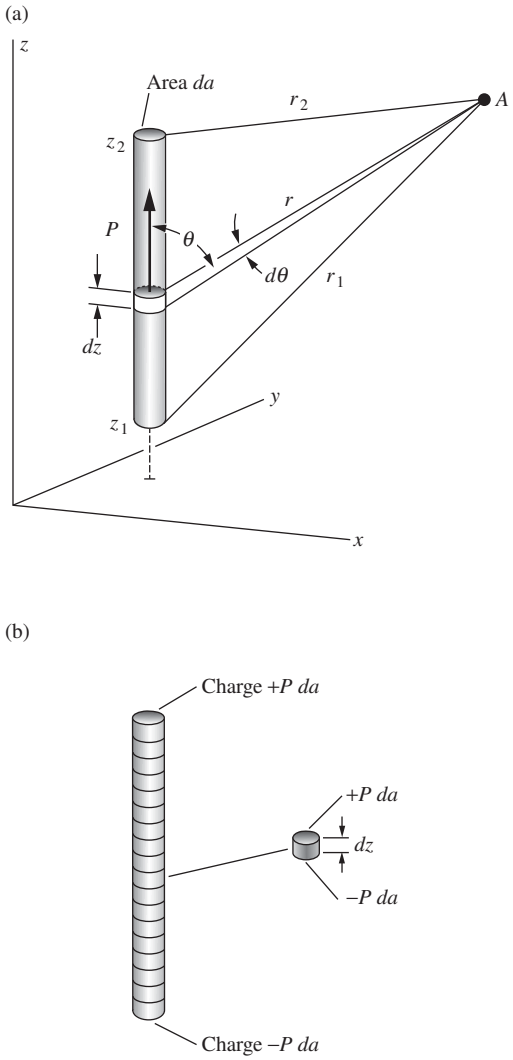

Figure 10.15.
A column of polarized material (a) produces the same field, at an external point A, as two charges, one at each end of the column (b).

interested only in the electric field *they* produce when they are in this condition; later we can introduce any fields from other sources that might be around. If you like, you can imagine that these are molecules with permanent dipole moments that have been lined up neatly, all pointing the same way, and frozen in position. All we need to specify is N, the number of dipoles per cubic meter, and the moment of each dipole $\mathbf{p}$. We shall assume that N is so large that any macroscopically small volume dv contains quite a large number of dipoles. The total dipole strength in such a volume is $\mathbf{p}N\,dv$. At any point far away from this volume element compared with its size, the electric field from these particular dipoles would be practically the same if they were replaced by a single dipole moment of strength $\mathbf{p}N\,dv$. We shall call $\mathbf{p}N$ the density of polarization, and denote it by $\mathbf{P}$, a vector quantity with the units C-m/m^3 (or C/m^2):

$$\mathbf{P} \equiv \mathbf{p}N = \frac{\text{dipole moment}}{\text{volume}}. \tag{10.31}$$

$\mathbf{P}\,dv$ is the dipole moment to be associated with any small-volume element dv for the purpose of computing the electric field at a distance. By the way, our matter has been assembled from neutral molecules only; there is no net charge in the system or on any molecule, so we have *only* the dipole moments to consider as sources of a distant field.

Figure 10.15 shows a slender column, or cylinder, of this polarized material. Its cross section is da, and it extends vertically from z_1 to z_2. The polarization density $\mathbf{P}$ within the column is uniform over the length and points in the positive z direction. We are about to calculate the electric potential, at some external point, due to this column of polarization. An element of the cylinder, of height dz, has a dipole moment $\mathbf{P}\,dv = \mathbf{P}\,da\,dz$. Its contribution to the potential at the point A can be written down by referring back to our formula Eq. (10.15) for the potential of a dipole, that is,

$$d\phi_A = \frac{P\,da\,dz\cos\theta}{4\pi\epsilon_0 r^2}. \tag{10.32}$$

The potential due to the entire column is

$$\phi_A = \frac{P\,da}{4\pi\epsilon_0}\int_{z_1}^{z_2}\frac{dz\cos\theta}{r^2}. \tag{10.33}$$

This is simpler than it looks: $dz\cos\theta$ is just $-dr$, so that the integrand is a perfect differential, $d(1/r)$. The result of the integration is then

$$\boxed{\phi_A = \frac{P\,da}{4\pi\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \tag{10.34}$$

Equation (10.34) is precisely the same as the expression for the potential at A that would be produced by two point charges, a positive charge of magnitude $P\,da$ sitting on top of the column at a distance r_2 from A, and a negative charge of the same magnitude at the bottom of

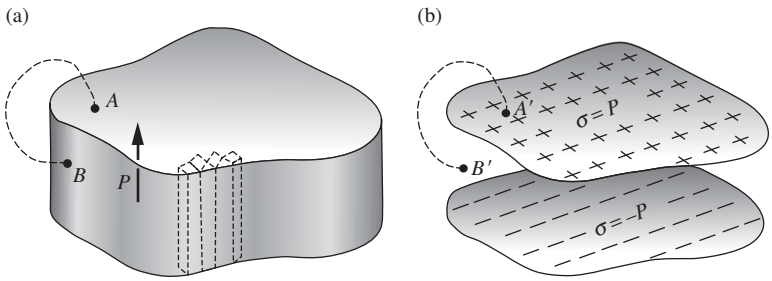

Figure 10.16.
A block of polarized material (a) is equivalent to two sheets of charge (b), as far as the field outside is concerned.

the column. The source consisting of a column of uniformly polarized matter is equivalent, at least so far as its field at all *external* points is concerned, to two concentrated charges. Note that nowhere have we assumed that A is far away from the column, that is, that r_1 and r_2 are much larger than the height of the column, $z_2 - z_1$. All that is required is that the distance from A to any point in the column is much larger than the size of the dipoles (assumed to be very small) and also much larger than the width of the column (also assumed to be small), for then Eq. (10.32) will be valid.

We can prove Eq. (10.34) in another way without any mathematics. Consider a small section of the column of height dz, containing a dipole moment $P\,da\,dz$. Let us make an imitation or substitute for this by taking an unpolarized insulator of the same size and shape and sticking a charge $P\,da$ on top of it and a charge $-P\,da$ on the bottom. This little block now has the same dipole moment as that bit of our original column, and therefore it will make an identical contribution to the field at any remote point A. (The field inside our substitute, or very close to it, may be different from the field of the original – we don't care about that.) Now make a whole set of such blocks and stack them up to imitate the polarized column; see Fig. 10.15(b). They must give the same field at A as the whole column does, for each block gave the same contribution as its counterpart in the original. Now see what we have! At every joint the positive charge on the top of one block coincides with the negative charge on the bottom of the block above it, making the charge equal zero. The only charges left uncompensated are the negative charge $-P\,da$ on the bottom of the bottom block and the positive charge $P\,da$ on the top of the top block. Seen from a distant point such as A ("distant" compared with the size of a block, not necessarily the whole column), these look like point charges. We conclude, as before, that two such charges produce at A exactly the same field as does our whole column of polarized material.

With no further calculation we can extend this to a slab, or right cylinder, of any proportions uniformly polarized in a direction perpendicular to its parallel faces; see Fig. 10.16(a). The slab can simply be subdivided into a bundle of columns, and the potential outside will be the sum of the contributions of the columns, each of which can be replaced

by a charge at either end. The charges on the top, $P\,da$ on each column end of area da, make up a uniform sheet of surface charge of density

$$\boxed{\sigma = P} \qquad \text{(in coulomb/meter}^2\text{)}. \tag{10.35}$$

We conclude that the potential everywhere *outside* a uniformly polarized slab or cylinder (not necessarily far away) is precisely what would result from two sheets of surface charge located where the top and bottom surfaces of the slab were located, carrying the constant surface charge density $\sigma = P$ and $\sigma = -P$, respectively; see Fig. 10.16(b).

We are not quite ready to say anything about the field *inside* the slab. However, we do know the potential at all points on the surface of the slab – top, bottom, or sides. Any two such points, A and B, can be connected by a path running entirely through the external field, so that the line integral $\int \mathbf{E} \cdot d\mathbf{s}$ is entirely determined by the external field. It must be the same as the integral along the path $A'B'$ in Fig. 10.16(b). A point literally on the surface of the dielectric might be within range of the intense molecular fields, the near field of the molecule that we have left out of our account. Let's agree to define the boundary of the dielectric as a surface far enough out from the outermost atomic nucleus – 10 or 20 angstroms would be margin enough – so that at any point outside this boundary, the near fields of the individual atoms make a negligible contribution to the whole line integral from A to B.

With this in mind, let's look at a rather thin, wide plate of polarized material, of thickness t, shown in cross section in Fig. 10.17(a). Figure 10.17(b) shows, likewise in cross section, the equivalent sheets of charge. For the system of two charge sheets, we know the field, of course, in the space both outside and between the sheets. The field strength inside, well away from the edges, must be just σ/ϵ_0, pointing down, and the potential difference between points A' and B' is therefore $\sigma t/\epsilon_0$. The *same potential difference* must exist between corresponding points A and B on our polarized slab, because the entire *external* field is the same in the two systems.

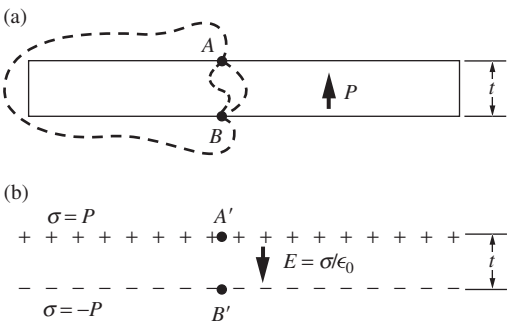

Figure 10.17.
(a) The line integral of $\mathbf{E}$ from A to B must be the same over all paths, internal or external, because the internal microscopic or atomic electric fields also are conservative (curl $\mathbf{E} = 0$). The equivalent charge sheets (b) have the same external field.

10.7.2 The field inside matter

We can now address the field inside polarized matter. Is the internal field the same in the two systems in Fig. 10.17? Certainly *not,* because the slab is full of positive nuclei and electrons, with fields on the order of 10^{11} volts per meter pointing in one direction here, another direction there. But one thing *is* the same: the line integral of the field, reckoned over *any* internal path from A to B, must be just $\phi_B - \phi_A$, which, as we have seen, is the same as $\phi_{B'} - \phi_{A'}$, which is equal to $\sigma t/\epsilon_0$, or Pt/ϵ_0. This must be so because the introduction of atomic charges, no matter what their distribution, cannot destroy the conservative property of the electric field, expressed in the statement that $\int \mathbf{E} \cdot d\mathbf{s}$ is independent of path, or curl $\mathbf{E} = 0$.

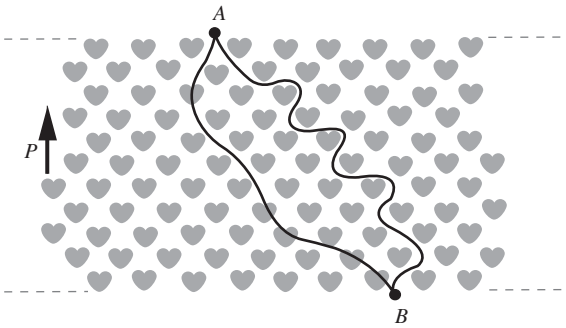

Figure 10.18.
Over any path from A to B, the line integral of the actual microscopic field is the same.

We know that in Fig. 10.17(b) the potential difference between the top and bottom sheets is nearly constant, except near the edges, because the interior electric field is practically uniform. Therefore in the central area of our polarized plate the potential difference between top and bottom must likewise be constant. In this region the line integral $\int_A^B \mathbf{E} \cdot d\mathbf{s}$ taken from *any* point A on top of the slab to *any* point B on the bottom, by *any* path, must always yield the same value Pt/ϵ_0. Figure 10.18 is a "magnified view" of the central region of the slab, in which the polarized molecules have been made to look something like H_2O molecules all pointing the same way. We have not attempted to depict the very intense fields that exist between and inside the molecules. (The field ten angstroms away from a water molecule is on the order of a hundred megavolts per meter, as you can discover from Fig. 10.14 and Eq. (10.18).) You must imagine some rather complicated field configurations in the neighborhood of each molecule. Now, the $\mathbf{E}$ in $\int \mathbf{E} \cdot d\mathbf{s}$ represents the *total electric field* at a given point in space, inside or outside a molecule; it includes these complicated and intense fields just mentioned. We have reached the remarkable conclusion that *any* path through this welter of charges and fields, whether it dodges molecules or penetrates them, must yield the same value for the path integral, namely the value we find in the system of Fig. 10.17(b), where the field is quite uniform and has the strength P/ϵ_0.

This tells us that the *spatial average* of the electric field within our polarized slab must be $-\mathbf{P}/\epsilon_0$. By the spatial average of a field $\mathbf{E}$ over some volume V, which we might denote by $\langle \mathbf{E} \rangle_V$, we mean precisely this:

$$\langle \mathbf{E} \rangle_V = \frac{1}{V} \int_V \mathbf{E}\, dv. \tag{10.36}$$

One way to sample impartially the field in many equal volumes dv into which V might be divided would be to measure the field along each line in a bundle of closely spaced parallel lines. We have just seen that the line integral of $\mathbf{E}$ along any or all such paths is the same as if we were in a constant electric field of strength $-\mathbf{P}/\epsilon_0$. That is the justification for the conclusion that, within the polarized dielectric slab of Figs. 10.17 and

10.18, the spatial average of the field due to all the charges that belong to the dielectric is

$$\boxed{\langle \mathbf{E} \rangle = -\frac{\mathbf{P}}{\epsilon_0}} \tag{10.37}$$

This average field is a *macroscopic* quantity. The volume over which we take the average should be large enough to include very many molecules, otherwise the average will fluctuate from one such volume to the adjoining one. The average field $\langle \mathbf{E} \rangle$ defined by Eq. (10.36) is really the only kind of *macroscopic* electric field in the interior of a dielectric that we can talk about. It provides the only satisfactory answer, in the context of a macroscopic description of matter, to the question, What is the electric field inside a dielectric material?

We may call the $\mathbf{E}$ in the integrand on the right, in Eq. (10.36), the *microscopic* field. If we imagine that we could measure the field values we need for the path integral, we will be measuring electric fields in vacuum, in the presence, of course, of electric charge. We will need very tiny instruments, for we may be called on to measure the field at a particular point just inside one end of a certain molecule. Have we any right to talk in this way about taking the line integral of $\mathbf{E}$ along some path that skirts the southwest corner of a particular molecule and then tunnels through its neighbor? Yes. The justification is the massive evidence that the laws of electromagnetism work down to a scale of distances much smaller than atomic size. We can even describe an experiment that would serve to measure the average of the microscopic electric field along a path defined well within the limits of atomic dimensions. All we have to do is shoot an energetic charged particle, an alpha particle for example, through the material. From the net change in its momentum, the average electric field that acted on it, over its whole path, could be inferred.

Let us review the properties of the average, or macroscopic, field $\langle \mathbf{E} \rangle$ defined by Eq. (10.36). Its line integral $\int_A^B \langle \mathbf{E} \rangle \cdot d\mathbf{s}$ between any two points A and B that are reasonably far apart is independent of the path. It follows that curl $\langle \mathbf{E} \rangle = 0$ and that $\langle \mathbf{E} \rangle$ is the negative gradient of a potential $\langle \phi \rangle$. This potential function $\langle \phi \rangle$ is itself a smoothed-out average, in the sense of Eq. (10.36), of the microscopic potential ϕ. (The latter rises to several million volts in the interior of every atomic nucleus!) The surface integral of $\langle \mathbf{E} \rangle$, $\int \langle \mathbf{E} \rangle \cdot d\mathbf{a}$, over any surface that encloses a reasonably large volume, is equal to $1/\epsilon_0$ times the charge within that volume.[3] That is to say, $\langle \mathbf{E} \rangle$ obeys Gauss's law, a statement we can also make in differential

[3] We state this without proof, postponing consideration of the relation of the surface integral of an average field to the average of surface integrals of the microscopic field to Chapter 11, where the question arises in Section 11.8 in connection with the magnetic field inside matter. (See Fig. 11.18.)

form: div $\langle \mathbf{E} \rangle = \langle \rho \rangle / \epsilon_0$, with the understanding that $\langle \rho \rangle$ too is a local average over a suitably macroscopic volume. In short, the spatial average quantities $\langle \mathbf{E} \rangle$, $\langle \phi \rangle$, and $\langle \rho \rangle$ are related to one another in the same way as are the microscopic electric field, potential, and charge density in vacuum.

From now on, when we speak of the electric field $\mathbf{E}$ inside any piece of matter much larger than a molecule, we will mean an average, or macroscopic, field as defined by Eq. (10.36), even when the brackets $\langle\ \rangle$ are omitted.

10.8 Another look at the capacitor

At the beginning of this chapter we explained in a qualitative way how the presence of a dielectric between the plates of a capacitor increases its capacitance. Now we are ready to analyze quantitatively the dielectric-filled capacitor. What we have just learned about the electric field inside matter is the key to the problem. We identified as the macroscopic field $\mathbf{E}$, the spatial average of the microscopic field. The line integral of that macroscopic $\mathbf{E}$ between any two points A and B is path-independent and equal to the potential difference. Looking back at Fig. 10.2(a) we observe that the field $\mathbf{E}$ in the empty capacitor must have had the value ϕ_{12}/s. But the potential difference between the plates, ϕ_{12}, which was established by the battery, was exactly the same in the dielectric-filled capacitor in Fig. 10.2(b). Hence the field $\mathbf{E}$ in the dielectric, understood now as the macroscopic field, must have had the same value too, for it extends and is uniform over the same distance s. (The layers in the diagram are actually negligible in thickness compared with s.)

The fact that the $\mathbf{E}$ fields are the same implies that the total charge on and near the top plate in the dielectric-filled capacitor must be the same as it was in the empty capacitor, namely Q_0. To prove that, we need only invoke Gauss's law for a suitable imaginary box enclosing the charge layers, as indicated in Fig. 10.19. Now, the charge is made up of two parts, the charge on the plate Q (which will flow off when the capacitor is discharged) and Q', the charge that belongs to the dielectric. The charge on the plate is given by $Q = \kappa Q_0$. That was our definition of κ. Therefore, if $Q + Q' = Q_0$ as we have just concluded, we must have

$$Q' = Q_0 - Q = Q_0(1 - \kappa). \tag{10.38}$$

We can think of this system as the superposition of a vacuum capacitor and a polarized dielectric slab, Fig. 10.19(b) and (c). In the vacuum capacitor with charge κQ_0, the electric field E'' would be κ times the field E. In the isolated polarized dielectric slab the field E' is $-P/\epsilon_0$, as stated in Eq. (10.37). The superposition of these two objects creates the actual field E. That is,

$$E = E'' + E' = \kappa E - \frac{P}{\epsilon_0}, \tag{10.39}$$

CHAPTER

4 Electric Fields in Matter

4.1 ■ POLARIZATION

4.1.1 ■ Dielectrics

In this chapter, we shall study electric fields in matter. Matter, of course, comes in many varieties—solids, liquids, gases, metals, woods, glasses—and these substances do not all respond in the same way to electrostatic fields. Nevertheless, *most* everyday objects belong (at least, in good approximation) to one of two large classes: **conductors** and **insulators** (or **dielectrics**). We have already talked about conductors; these are substances that contain an "unlimited" supply of charges that are free to move about through the material. In practice, what this ordinarily means is that many of the electrons (one or two per atom, in a typical metal) are not associated with any particular nucleus, but roam around at will. In dielectrics, by contrast, *all charges are attached to specific atoms or molecules*—they're on a tight leash, and all they can do is move a bit *within* the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials. There are actually *two* principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: *stretching* and *rotating*. In the next two sections I'll discuss these processes.

4.1.2 ■ Induced Dipoles

What happens to a neutral atom when it is placed in an electric field **E**? Your first guess might well be: "Absolutely nothing—since the atom is not charged, the field has no effect on it." But that is incorrect. Although the atom as a whole is electrically neutral, there *is* a positively charged core (the nucleus) and a negatively charged electron cloud surrounding it. These two regions of charge within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way. In principle, if the field is large enough, it can pull the atom apart completely, "ionizing" it (the substance then becomes a conductor). With less extreme fields, however, an equilibrium is soon established, for if the center of the electron cloud does not coincide with the nucleus, these positive and negative charges attract one another, and that holds the atom together. The two opposing forces—**E** pulling the electrons and nucleus apart, their mutual attraction drawing them back together—reach a balance, leaving the

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Data from: Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, 2010).

atom **polarized,** with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment **p**, which points in the *same direction as* **E**. Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\mathbf{p} = \alpha \mathbf{E}. \tag{4.1}$$

The constant of proportionality α is called **atomic polarizability.** Its value depends on the detailed structure of the atom in question. Table 4.1 lists some experimentally determined atomic polarizabilities.

Example 4.1. A primitive model for an atom consists of a point nucleus $(+q)$ surrounded by a uniformly charged spherical cloud $(-q)$ of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

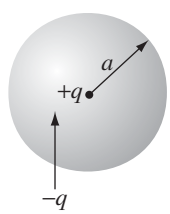

FIGURE 4.1

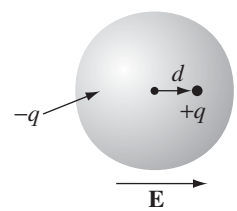

FIGURE 4.2

Solution

In the presence of an external field **E**, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Fig. 4.2. (Because the actual displacements involved are extremely small, as you'll see in Prob. 4.1, it is reasonable to assume that the electron cloud retains its spherical shape.) Say that equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point, the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$, where E_e is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

(Prob. 2.12). At equilibrium, then,

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \quad \text{or} \quad p = qd = (4\pi\epsilon_0 a^3)E.$$

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v, \tag{4.2}$$

where v is the volume of the atom. Although this atomic model is extremely crude, the result (Eq. 4.2) is not too bad—it's accurate to within a factor of four or so for many simple atoms.

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than in others. Carbon dioxide (Fig. 4.3), for instance, has a polarizability of 4.5×10^{-40} C^2·m/N when you apply the field along the axis of the molecule, but only 2×10^{-40} for fields perpendicular to this direction. When the field is at some *angle* to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_\perp \mathbf{E}_\perp + \alpha_\parallel \mathbf{E}_\parallel.$$

In this case, the induced dipole moment may not even be in the same *direction* as **E**. And CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line; for a completely asymmetrical molecule, Eq. 4.1 is replaced by the most general linear relation between **E** and **p**:

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \tag{4.3}$$

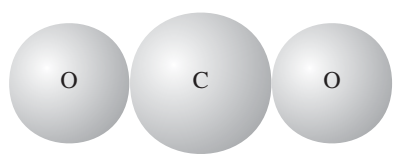

FIGURE 4.3

The set of nine constants α_{ij} constitute the **polarizability tensor** for the molecule. Their values depend on the orientation of the axes you use, though it is always possible to choose "principal" axes such that all the off-diagonal terms (α_{xy}, α_{zx}, etc.) vanish, leaving just three nonzero polarizabilities: α_{xx}, α_{yy}, and α_{zz}.

Problem 4.1 A hydrogen atom (with the Bohr radius of half an angstrom) is situated between two metal plates 1 mm apart, which are connected to opposite terminals of a 500 V battery. What fraction of the atomic radius does the separation distance d amount to, roughly? Estimate the voltage you would need with this apparatus to ionize the atom. [Use the value of α in Table 4.1. *Moral:* The displacements we're talking about are *minute,* even on an atomic scale.]

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint:* First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]

Problem 4.3 According to Eq. 4.1, the induced dipole moment of an atom is proportional to the external field. This is a "rule of thumb," not a fundamental law, and it is easy to concoct exceptions—in theory. Suppose, for example, the charge density of the electron cloud were proportional to the distance from the center, out to a radius R. To what power of E would p be proportional in that case? Find the condition on $\rho(r)$ such that Eq. 4.1 will hold in the weak-field limit.

Problem 4.4 A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

4.1.3 ■ Alignment of Polar Molecules

The neutral atom discussed in Sect. 4.1.2 had no dipole moment to start with—**p** was *induced* by the applied field. Some molecules have built-in, permanent dipole moments. In the water molecule, for example, the electrons tend to cluster around the oxygen atom (Fig. 4.4), and since the molecule is bent at 105°, this leaves a negative charge at the vertex and a net positive charge on the opposite side. (The dipole moment of water is unusually large: 6.1×10^{-30} C·m; in fact, this is what accounts for its effectiveness as a solvent.) What happens when such molecules (called **polar molecules**) are placed in an electric field?

If the field is uniform, the *force* on the positive end, $\mathbf{F}_+ = q\mathbf{E}$, exactly cancels the force on the negative end, $\mathbf{F}_- = -q\mathbf{E}$ (Fig. 4.5). However, there will be a *torque:*

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.\end{aligned}$$

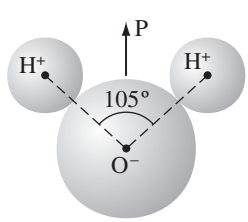

FIGURE 4.4

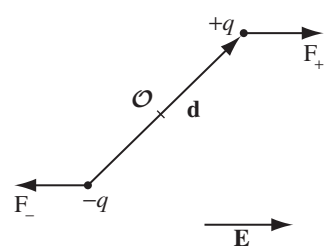

FIGURE 4.5

[1]For a more sophisticated approach, see W. A. Bowers, *Am. J. Phys.* **54,** 347 (1986).

Thus a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}.} \tag{4.4}$$

Notice that $\mathbf{N}$ is in such a direction as to line $\mathbf{p}$ up *parallel* to $\mathbf{E}$; a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is *non*uniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net *force* on the dipole, in addition to the torque. Of course, $\mathbf{E}$ must change rather abruptly for there to be significant variation in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics. Nevertheless, the formula for the force on a dipole in a nonuniform field is of some interest:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}),$$

where $\Delta\mathbf{E}$ represents the difference between the field at the plus end and the field at the minus end. Assuming the dipole is very short, we may use Eq. 1.35 to approximate the small change in E_x:

$$\Delta E_x \equiv (\nabla E_x) \cdot \mathbf{d},$$

with corresponding formulas for E_y and E_z. More compactly,

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E},$$

and therefore[2]

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.} \tag{4.5}$$

For a "perfect" dipole of infinitesimal length, Eq. 4.4 gives the torque *about the center of the dipole* even in a *non*uniform field; about any *other* point $\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$.

Problem 4.5 In Fig. 4.6, $\mathbf{p}_1$ and $\mathbf{p}_2$ are (perfect) dipoles a distance r apart. What is the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$? What is the torque on $\mathbf{p}_2$ due to $\mathbf{p}_1$? [In each case, I want the torque on the dipole *about its own center*. If it bothers you that the answers are not equal and opposite, see Prob. 4.29.]

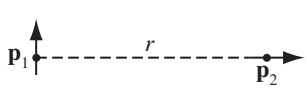

FIGURE 4.6

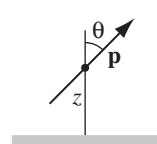

FIGURE 4.7

[2]In the present context, Eq. 4.5 could be written more conveniently as $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$. However, it is safer to stick with $(\mathbf{p} \cdot \nabla)\mathbf{E}$, because we will be applying the formula to materials in which the dipole moment (per unit volume) is itself a function of position and this second expression would imply (incorrectly) that $\mathbf{p}$ *too* is to be differentiated.

Problem 4.6 A (perfect) dipole **p** is situated a distance z above an infinite grounded conducting plane (Fig. 4.7). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on **p**. If the dipole is free to rotate, in what orientation will it come to rest?

Problem 4.7 Show that the energy of an ideal dipole **p** in an electric field **E** is given by

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E}.} \tag{4.6}$$

Problem 4.8 Show that the interaction energy of two dipoles separated by a displacement **r** is

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]. \tag{4.7}$$

[*Hint:* Use Prob. 4.7 and Eq. 3.104.]

Problem 4.9 A dipole **p** is a distance r from a point charge q, and oriented so that **p** makes an angle θ with the vector **r** from q to **p**.

(a) What is the force on **p**?

(b) What is the force on q?

4.1.4 ■ Polarization

In the previous two sections, we have considered the effect of an external electric field on an individual atom or molecule. We are now in a position to answer (qualitatively) the original question: What happens to a piece of dielectric material when it is placed in an electric field? If the substance consists of neutral atoms (or nonpolar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field.[3] If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motions compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed.)

Notice that these two mechanisms produce the same basic result: *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**. A convenient measure of this effect is

$$\mathbf{P} \equiv \textit{ dipole moment per unit volume,}$$

which is called the **polarization**. From now on we shall not worry much about how the polarization *got* there. Actually, the two mechanisms I described are not as clear-cut as I tried to pretend. Even in polar molecules there will be

[3]In asymmetric molecules, the induced dipole moment may not be parallel to the field, but if the molecules are randomly oriented, the perpendicular contributions will *average* to zero. Within a single crystal, the orientations are certainly *not* random, and we would have to treat this case separately.

some polarization by displacement (though generally it is a lot easier to rotate a molecule than to stretch it, so the second mechanism dominates). It's even possible in some materials to "freeze in" polarization, so that it persists after the field is removed. But let's forget for a moment about the *cause* of the polarization, and let's study the field that a chunk of polarized material *itself* produces. Then, in Sect. 4.3, we'll put it all together: the original field, which was *responsible* for **P**, plus the new field, which is *due* to **P**.

4.2 ■ THE FIELD OF A POLARIZED OBJECT

4.2.1 ■ Bound Charges

Suppose we have a piece of polarized material—that is, an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume **P** is given. *Question:* What is the field produced by this object (not the field that may have *caused* the polarization, but the field the polarization *itself* causes)? Well, we know what the field of an individual dipole looks like, so why not chop the material up into infinitesimal dipoles and integrate to get the total? As usual, it's easier to work with the potential. For a single dipole **p** (Eq. 3.99),

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\imath}}}{\imath^2}, \tag{4.8}$$

where $\boldsymbol{\imath}$ is the vector from the dipole to the point at which we are evaluating the potential (Fig. 4.8). In the present context, we have a dipole moment $\mathbf{p} = \mathbf{P}\,d\tau'$ in each volume element $d\tau'$, so the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau'. \tag{4.9}$$

That *does* it, in principle. But a little sleight-of-hand casts this integral into a much more illuminating form. Observing that

$$\boldsymbol{\nabla}'\left(\frac{1}{\imath}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2},$$

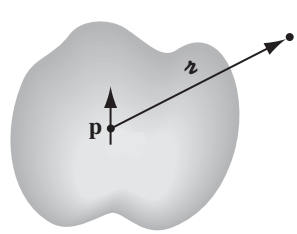

FIGURE 4.8

where (unlike Prob. 1.13) the differentiation is with respect to the *source* coordinates ($\mathbf{r}'$), we have

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \boldsymbol{\nabla}' \left(\frac{1}{\imath}\right) d\tau'.$$

Integrating by parts, using product rule number 5 (in the front cover), gives

$$V = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{V}} \boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{P}}{\imath}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\imath} (\boldsymbol{\nabla}' \cdot \mathbf{P})\, d\tau' \right],$$

or, invoking the divergence theorem,

$$V = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{\imath} \mathbf{P} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{1}{\imath} (\boldsymbol{\nabla}' \cdot \mathbf{P})\, d\tau'. \tag{4.10}$$

The first term looks like the potential of a surface charge

$$\boxed{\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}} \tag{4.11}$$

(where $\hat{\mathbf{n}}$ is the normal unit vector), while the second term looks like the potential of a volume charge

$$\boxed{\rho_b \equiv -\boldsymbol{\nabla} \cdot \mathbf{P}.} \tag{4.12}$$

With these definitions, Eq. 4.10 becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\imath}\, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\imath}\, d\tau'. \tag{4.13}$$

What this means is that the potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. 4.9, we could first find those **bound charges**, and then calculate the fields *they* produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law).

Example 4.2. Find the electric field produced by a uniformly polarized sphere of radius R.

Solution

We may as well choose the z axis to coincide with the direction of polarization (Fig. 4.9). The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform, but

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta,$$

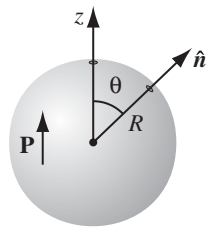

FIGURE 4.9

where θ is the usual spherical coordinate. What we want, then, is the field produced by a charge density $P\cos\theta$ plastered over the surface of a sphere. But we already computed the potential of such a configuration, in Ex. 3.9:

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for} \quad r < R. \tag{4.14}$$

This remarkable result will be very useful in what follows. Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for} \quad r \ge R, \tag{4.15}$$

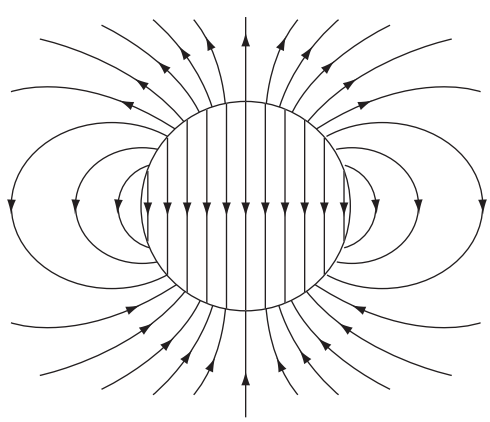

FIGURE 4.10

whose dipole moment is, not surprisingly, equal to the total dipole moment of the sphere:

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}. \tag{4.16}$$

The field of the uniformly polarized sphere is shown in Fig. 4.10.

Problem 4.10 A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

Problem 4.11 A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$, (ii) for $L \ll a$, and (iii) for $L \approx a$. [This is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

Problem 4.12 Calculate the potential of a uniformly polarized sphere (Ex. 4.2) directly from Eq. 4.9.

4.2.2 ■ Physical Interpretation of Bound Charges

In the last section we found that the field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges," σ_b and ρ_b. But this conclusion emerged in the course of abstract manipulations on the integral in Eq. 4.9, and left us with no clue as to the physical meaning of these bound charges. Indeed, some authors give you the impression that bound charges are in some sense "fictitious"—mere bookkeeping devices used to facilitate the calculation of fields. Nothing could be further from the truth: ρ_b and σ_b represent *perfectly genuine accumulations of charge.* In this section I'll explain how polarization leads to these charge distributions.

The basic idea is very simple: Suppose we have a long string of dipoles, as shown in Fig. 4.11. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over: plus at the right end and minus at the left. It is as if we had peeled off an electron at one end and carried it all the way down to the other end, though in fact no single electron made the whole trip—a lot of tiny displacements add up to one large one. We call the net charge at the ends a *bound* charge to remind ourselves that it cannot be removed;

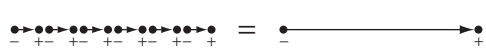

FIGURE 4.11

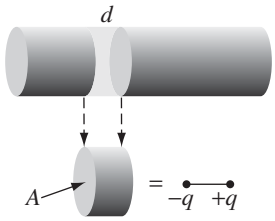

FIGURE 4.12

FIGURE 4.13

in a dielectric every electron is attached to a specific atom or molecule. But apart from that, bound charge is no different from any other kind.

To calculate the actual *amount* of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to **P**. The dipole moment of the tiny chunk shown in Fig. 4.12 is $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, this same dipole moment can be written qd. The bound charge that piles up at the right end of the tube is therefore

$$q = PA.$$

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P.$$

For an oblique cut (Fig. 4.13), the *charge* is still the same, but $A = A_{\text{end}} \cos\theta$, so

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}.$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ over the surface of the material. This is exactly what we found by more rigorous means in Sect. 4.2.1. But now we know where the bound charge *comes* from.

If the polarization is nonuniform, we get accumulations of bound charge *within* the material, as well as on the surface. A glance at Fig. 4.14 suggests that a diverging **P** results in a pileup of negative charge. Indeed, the net bound charge $\int \rho_b\, d\tau$

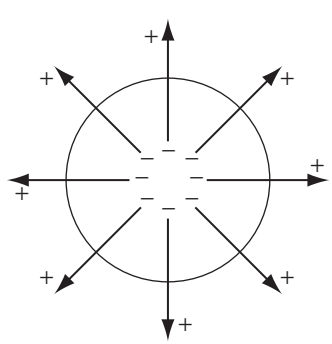

FIGURE 4.14

in a given volume is equal and opposite to the amount that has been pushed out through the surface. The latter (by the same reasoning we used before) is $\mathbf{P} \cdot \hat{\mathbf{n}}$ per unit area, so

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{P}) \, d\tau.$$

Since this is true for *any* volume, we have

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P},$$

confirming, again, the more rigorous conclusion of Sect. 4.2.1.

Example 4.3. There is another way of analyzing the uniformly polarized sphere (Ex. 4.2), which nicely illustrates the idea of a bound charge. What we have, really, is *two* spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the z direction), and all the minus charges move slightly *downward* (Fig. 4.15). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.

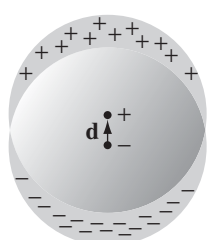

FIGURE 4.15

In Prob. 2.18, you calculated the field in the region of overlap between two uniformly charged spheres; the answer was

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \frac{q\mathbf{d}}{R^3},$$

where q is the total charge of the positive sphere, $\mathbf{d}$ is the vector from the negative center to the positive center, and R is the radius of the sphere. We can express this in terms of the polarization of the sphere, $\mathbf{p} = q\mathbf{d} = (\frac{4}{3}\pi R^3)\mathbf{P}$, as

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

Meanwhile, for points *outside,* it is as though all the charge on each sphere were concentrated at the respective center. We have, then, a dipole, with potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.$$

(Remember that **d** is some small fraction of an atomic radius; Fig. 4.15 is grossly exaggerated.) These answers agree, of course, with the results of Ex. 4.2.

Problem 4.13 A very long cylinder, of radius a, carries a uniform polarization **P** perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

[*Careful:* I said "uniform," not "radial"!]

Problem 4.14 When you polarize a neutral dielectric, the charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

4.2.3 ■ The Field Inside a Dielectric[4]

I have been sloppy about the distinction between "pure" dipoles and "physical" dipoles. In developing the theory of bound charges, I assumed we were working with the pure kind—indeed, I started with Eq. 4.8, the formula for the potential of a perfect dipole. And yet, an actual polarized dielectric consists of *physical* dipoles, albeit extremely tiny ones. What is more, I presumed to represent discrete molecular dipoles by a continuous density function **P**. How can I justify this method? *Outside* the dielectric there is no real problem: here we are far away from the molecules ($\mathcal{r}$ is many times greater than the separation distance between plus and minus charges), so the dipole potential dominates overwhelmingly and the detailed "graininess" of the source is blurred by distance. *Inside* the dielectric, however, we can hardly pretend to be far from all the dipoles, and the procedure I used in Sect. 4.2.1 is open to serious challenge.

In fact, when you stop to think about it, the electric field inside matter must be fantastically complicated, on the microscopic level. If you happen to be very near an electron, the field is gigantic, whereas a short distance away it may be small or may point in a totally different direction. Moreover, an instant later, as the atoms move about, the field will have altered entirely. This true **microscopic** field would be utterly impossible to calculate, nor would it be of much interest if you could. Just as, for macroscopic purposes, we regard water as a continuous fluid, ignoring its molecular structure, so also we can ignore the microscopic

[4]This section can be skipped without loss of continuity.

bumps and wrinkles in the electric field inside matter, and concentrate on the **macroscopic** field. This is defined as the *average* field over regions large enough to contain many thousands of atoms (so that the uninteresting microscopic fluctuations are smoothed over), and yet small enough to ensure that we do not wash out any significant large-scale variations in the field. (In practice, this means we must average over regions much smaller than the dimensions of the object itself.) Ordinarily, the macroscopic field is what people *mean* when they speak of "the" field inside matter.[5]

It remains to show that the macroscopic field is what we actually obtain when we use the methods of Sect. 4.2.1. The argument is subtle, so hang on. Suppose I want to calculate the macroscopic field at some point $\mathbf{r}$ within a dielectric (Fig. 4.16). I know I must average the true (microscopic) field over an appropriate volume, so let me draw a small sphere about $\mathbf{r}$, of radius, say, a thousand times the size of a molecule. The macroscopic field at $\mathbf{r}$, then, consists of two parts: the average field over the sphere due to all charges *outside*, plus the average due to all charges *inside:*

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}.$$

You proved in Prob. 3.47(d) that the average field (over a sphere), produced by charges *outside*, is equal to the field they produce at the center, so $\mathbf{E}_{\text{out}}$ is the field at $\mathbf{r}$ due to the dipoles exterior to the sphere. These are far enough away that we can safely use Eq. 4.9:

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \int_{\text{outside}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\boldsymbol{\imath}}}{\imath^2} \, d\tau'. \tag{4.17}$$

The dipoles *inside* the sphere are too close to treat in this fashion. But fortunately all we need is their *average* field, and that, according to Eq. 3.105, is

$$\mathbf{E}_{\text{in}} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3},$$

regardless of the details of the charge distribution within the sphere. The only relevant quantity is the total dipole moment, $\mathbf{p} = (\frac{4}{3}\pi R^3)\,\mathbf{P}$:

$$\mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}. \tag{4.18}$$

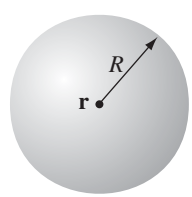

FIGURE 4.16

[5] In case the notion of macroscopic fields sounds suspicious to you, let me point out that you do *exactly* the same averaging whenever you speak of the *density* of a material.

Now, by assumption, the sphere is small enough that $\mathbf{P}$ does not vary significantly over its volume, so the term *left out* of the integral in Eq. 4.17 corresponds to the field at the center of a *uniformly* polarized sphere, to wit: $-(1/3\epsilon_0)\mathbf{P}$ (Eq. 4.14). But this is precisely what $\mathbf{E}_{\text{in}}$ (Eq. 4.18) puts back in! The macroscopic field, then, is given by the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau', \tag{4.19}$$

where the integral runs over the *entire* volume of the dielectric. This is, of course, what we used in Sect. 4.2.1; without realizing it, we were correctly calculating the averaged, macroscopic field, for points inside the dielectric.

You may have to reread the last couple of paragraphs for the argument to sink in. Notice that it all revolves around the curious fact that the average field over *any* sphere (due to the charge inside) is the same as the field at the center of a *uniformly polarized* sphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field. Incidentally, while the argument ostensibly relies on the spherical shape I chose to average over, the macroscopic field is certainly independent of the geometry of the averaging region, and this is reflected in the final answer, Eq. 4.19. Presumably one could reproduce the same argument for a cube or an ellipsoid or whatever—the calculation might be more difficult, but the conclusion would be the same.

4.3 ■ THE ELECTRIC DISPLACEMENT

4.3.1 ■ Gauss's Law in the Presence of Dielectrics

In Sect. 4.2 we found that the effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**, ρ_f). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f, \tag{4.20}$$

and Gauss's law reads

$$\epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla}\cdot\mathbf{P} + \rho_f,$$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

It is convenient to combine the two divergence terms:

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

The expression in parentheses, designated by the letter **D**,

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},} \tag{4.21}$$

is known as the **electric displacement**. In terms of **D**, Gauss's law reads

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,} \tag{4.22}$$

or, in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}, \tag{4.23}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges,* and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, by the mechanisms of Sect. 4.1, and this polarization produces the bound charge. In a typical problem, therefore, we know ρ_f, but we do not (initially) know ρ_b; Eq. 4.23 lets us go right to work with the information at hand. In particular, whenever the requisite symmetry is present, we can immediately calculate **D** by the standard Gauss's law methods.

Example 4.4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.

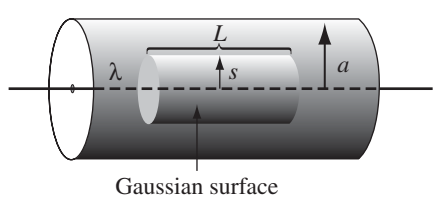

FIGURE 4.17

Solution

Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

It may appear to you that I left out the surface bound charge σ_b in deriving Eq. 4.22, and in a sense that is true. We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here ρ_b blows up,[6] taking the divergence of **E** with it. But everywhere *else* the logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness, within which the polarization tapers off to zero (probably a more realistic model than an abrupt cut-off anyway), then there *is* no surface bound charge; ρ_b varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied *everywhere.* At any rate, the integral form (Eq. 4.23) is free from this "defect."

Problem 4.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

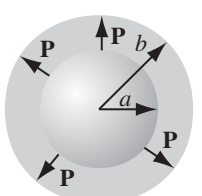

FIGURE 4.18

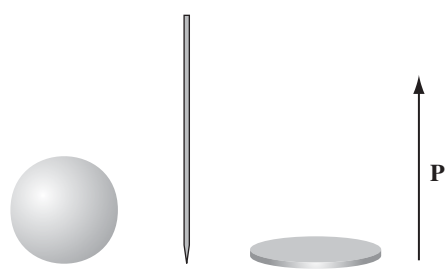

FIGURE 4.19

[6]The polarization drops abruptly to zero outside the material, so its *derivative* is a delta function (see Prob. 1.46). The surface bound charge *is* precisely this term—in this sense it is actually *included* in ρ_b, but we ordinarily prefer to handle it separately as σ_b.

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find **D**, and then get **E** from Eq. 4.21. [Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \epsilon_0\mathbf{E}_0 + \mathbf{P}$.

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and **P**. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and **P**. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

(b) Do the same for a long needle-shaped cavity running parallel to **P** (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to **P** (Fig. 4.19c).

Assume the cavities are small enough that **P**, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

4.3.2 ■ A Deceptive Parallel

Equation 4.22 looks just like Gauss's law, only the *total* charge density ρ is replaced by the *free* charge density ρ_f, and **D** is substituted for $\epsilon_0\mathbf{E}$. For this reason, you may be tempted to conclude that **D** is "just like" **E** (apart from the factor ϵ_0), except that its source is ρ_f instead of ρ: "To solve problems involving dielectrics, you just forget all about the bound charge—calculate the field as you ordinarily would, only call the answer **D** instead of **E**." This reasoning is seductive, but the conclusion is false; in particular, there is no "Coulomb's law" for **D**:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho_f(\mathbf{r}')\,d\tau'.$$

The parallel between **E** and **D** is more subtle than that.

For the divergence alone is insufficient to determine a vector field; you need to know the curl as well. One tends to forget this in the case of electrostatic fields because the curl of **E** is always zero. But the curl of **D** is *not* always zero.

$$\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}, \tag{4.25}$$

and there is no reason, in general, to suppose that the curl of **P** vanishes. Sometimes it does, as in Ex. 4.4 and Prob. 4.15, but more often it does not. The bar electret of Prob. 4.11 is a case in point: here there is no free charge anywhere, so if you really believe that the only source of **D** is ρ_f, you will be forced to conclude that $\mathbf{D} = \mathbf{0}$ everywhere, and hence that $\mathbf{E} = (-1/\epsilon_0)\mathbf{P}$ inside and $\mathbf{E} = \mathbf{0}$ outside the electret, which is obviously wrong. (I leave it for you to find the place where $\nabla \times \mathbf{P} \neq \mathbf{0}$ in this problem.) Because $\nabla \times \mathbf{D} \neq \mathbf{0}$, moreover, **D** cannot be expressed as the gradient of a scalar—there is no "potential" for **D**.

Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get **D** directly from Eq. 4.23 by the usual Gauss's law methods. (Evidently in such cases $\nabla \times \mathbf{P}$ is automatically zero, but since symmetry alone dictates the answer, you're not really obliged to worry about the curl.) If the requisite symmetry is absent, you'll have to think of another approach, and, in particular, you must *not* assume that **D** is determined exclusively by the free charge.

4.3.3 ■ Boundary Conditions

The electrostatic boundary conditions of Sect. 2.3.5 can be recast in terms of **D**. Equation 4.23 tells us the discontinuity in the component perpendicular to an interface:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f, \tag{4.26}$$

while Eq. 4.25 gives the discontinuity in parallel components:

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}. \tag{4.27}$$

In the presence of dielectrics, these are sometimes more useful than the corresponding boundary conditions on **E** (Eqs. 2.31 and 2.32):

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma, \tag{4.28}$$

and

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}. \tag{4.29}$$

You might try applying them, for example, to Probs. 4.16 and 4.17.

Problem 4.17 For the bar electret of Prob. 4.11, make three careful sketches: one of **P**, one of **E**, and one of **D**. Assume L is about $2a$. [*Hint:* **E** lines terminate on charges; **D** lines terminate on *free* charges.]

4.4 ■ LINEAR DIELECTRICS

4.4.1 ■ Susceptibility, Permittivity, Dielectric Constant

In Sects. 4.2 and 4.3 we did not commit ourselves as to the *cause* of **P**; we dealt only with the *effects* of polarization. From the qualitative discussion of Sect. 4.1, though, we know that the polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, in fact, the polarization is *proportional* to the field, provided **E** is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{4.30}$$